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A VARIATIONAL METHOD FOR SOLVING THE CONTACT PROBLEM IN THE THEORY OF ELASTICITY WITH FRICTION†

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The problem of the contact interaction of an elastic body with a rigid support with Coulomb friction in the contact area is considered. It is assumed that the contact area is known and does not change during loading, and the displacements and their gradients are small. The influence of tangential displacements on the contact pressure is investigated. The sufficient conditions for which no such effect occurs are formulated. The existence and uniqueness of a generalized solution of the variational formulation of the problem is proved using the operator of the influence of displacements on the contact pressure. © 1997 Elsevier Science Ltd. All rights reserved.

An incremental formulation of the contact problem for general boundary conditions on the contact surface and an attempt to prove the existence and uniqueness of the generalized solution are presented in [1]. Special cases of frictional contact problems, for which the existence and uniqueness of the generalized solution are proved only on the assumption that the value of the friction force at points of slip is independent of the solution (the problem with given normal pressure or friction) have also been considered [2–4]. Problems in which the contact surface and the pressure on it are independent of shear stresses have been investigated in [5–7]. A variational principle has been formulated for this class of problem and has been used to establish the existence and uniqueness of the solution. The special case of a problem with friction with no solution is given in [8]. The question of the existence and uniqueness of generalized solutions of problems with friction in a general formulation remains open, even for linearly-elastic materials.

Our purpose here is to continue to develop variational methods for investigating contact interaction allowing for friction forces, including an analysis of the existence and uniqueness of the generalized solution.

1. BASIC CONSTRAINTS ON THE FORMULATION OF THE PROBLEM. FORMULATION OF THE BOUNDARY CONDITIONS IN INCREMENTS ON THE CONTACT SURFACE

We will consider an approach to the solution of the elastic contact problem with Coulomb friction. Because of the difficulty of investigating the solvability of frictional problems [2, p. 148], we have made some simplifications, the main ones being the following: the contact region is known and does not change during loading, the displacements and their gradients are small, the trajectories of motion of points of the contact surface relative to the rigid support are rectilinear, and we consider loading in which the normal reaction of the rigid support is non-zero at each point.

The last constraint is associated with the fact that, when there is no normal reaction during loading, the directions of the relative displacements during loading are not known in advance and must be found from the solution of the problem. There is no difficulty in formulating the differential problem and constructing the variational principle without the last constraint (see [1] for example), but it is then impossible to prove the existence and uniqueness of the solution.

Note also that these assumptions give us a model problem. However, the possibility of extending the proposed method to the contact interaction of two deformable bodies can have practical applications for solving problems of temperature and press fittings.

The contact boundary conditions in increments will be formulated in the light of the above assumptions. At each point of the contact surface $\mathbf{L}_c$ Coulomb's law of friction is satisfied:

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in the slip region

$$\mathbf{v} \neq 0, \quad |\sigma_\tau| = \mathbf{f}|\sigma_\mathbf{n}|, \qquad \exists\lambda > 0: \; \sigma_\tau = -\lambda\mathbf{v} \tag{1.1}$$

in the adhesion region

$$\mathbf{v} = 0, \quad |\sigma_\tau| \leqslant \mathbf{f}|\sigma_\mathbf{n}| \tag{1.2}$$

where $\sigma_\mathbf{n}$, σ_τ are the projections of the reaction of the support onto the outer unit normal $\mathbf{n}$ and the tangent τ to the contact surface $\mathbf{L}_c$, the τ axis is in the direction of the trajectory of relative motion, $\mathbf{v}$ is the projection of the velocity of a point of the body relative to the rigid support onto the τ axis, and $\mathbf{f}(1 > \mathbf{f} > 0)$ is the friction factor. At points of the contact surface where the friction (or cohesive) force is non-zero, the tangent τ is defined by the condition that it is in the same direction as that force.

To take the loading history into account, we will seek the solution of the problem in increments, choosing the load in such a way that at each point of the contact surface $\mathbf{L}_c$, the increments of the components of the stress vector are of a higher order of smallness than its maximum possible tangential component at the time of loading, equal to $\mathbf{f}|\sigma_\mathbf{n}|$. This condition enables us to determine the direction of possible relative displacement (in a direction opposite to the positive direction of the tangent) at points of maximum equilibrium and at points where the reaction of the support is in a small neighbourhood of the surface of the cone of friction, and also eliminates the possibility of the contact surface separating from the support during loading.

Suppose that at some arbitrary time the contact surface comprises an adhesion area $\mathbf{L}_0 = \{\mathbf{x} \in \mathbf{L}_c \mid |\sigma_\tau| < \mathbf{f}|\sigma_\mathbf{n}|\}$ and the area $\mathbf{L}_1 = \{\mathbf{x} \in \mathbf{L}_c \mid |\sigma_\tau| = \mathbf{f}|\sigma_\mathbf{n}|\}$, which joins the slip surface ($\mathbf{v} \neq 0$) and the surface of limiting equilibrium ($\mathbf{v} = 0$). We then have $\mathbf{L}_c = \mathbf{L}_0 \cup \mathbf{L}_1$.

After loading, the contact surface can be represented as the union of two parts

$$\mathbf{L}_c = \mathbf{L}_0^* \cup \mathbf{L}_1^*, \quad \mathbf{L}_0^* = \{\mathbf{x} \in \mathbf{L}_c \mid \delta\mathbf{u} = 0\}, \quad \mathbf{L}_1^* = \{\mathbf{x} \in \mathbf{L}_c \mid \delta\mathbf{u} \neq 0\}$$

where $\delta\mathbf{u}$ are the displacements of points of the contact surface relative to the rigid support during loading. The region $\mathbf{L}_0^*$ (region $\mathbf{L}_1^*$) is formed by points of the contact surface at which adhesion occurs loading (shear has occurred).

We now assume that shear during loading takes place continuously. Then in the region $\mathbf{L}_1^*$

$$|\sigma_\tau + \delta\sigma_\tau| = \mathbf{f}\,|\sigma_\mathbf{n} + \delta\sigma_\mathbf{n}| \tag{1.3}$$

where $\delta\sigma_\tau$ and $\delta\sigma_\mathbf{n}$ are the increments of the tangential and normal components of reactions of the rigid support during loading. From Eqs (1.1) and (1.2) and the assumption about the shape of the trajectories of motion of points of the contact surface, we conclude that the relative displacements on the contact surface satisfy the conditions

$$\begin{aligned} &\delta\mathbf{u}_\mathbf{n} = 0, \quad \delta\mathbf{u}_\mathbf{b} = 0 \quad \text{on} \;\; \mathbf{L}_c \\ &\delta\mathbf{u}_\tau = 0 \;\; \text{on} \;\; \mathbf{L}_0^*, \quad \delta\mathbf{u}_\tau < 0 \;\; \text{on} \;\; \mathbf{L}_1^* \end{aligned} \tag{1.4}$$

$\delta\mathbf{u}_\mathbf{b} = 0$ is the projection of the displacement onto an axis perpendicular to the normal n and the tangent τ. Finally, using relations (1.3) and (1.4), the boundary conditions can be formulated as follows:

$$\begin{aligned} &\delta\mathbf{u}_\mathbf{n} = 0, \quad \delta\mathbf{u}_\mathbf{b} = 0 \\ &\delta\mathbf{u}_\tau = 0 \Rightarrow \sigma_\tau + \delta\sigma_\tau + \mathbf{f}(\sigma_\mathbf{n} + \delta\sigma_\mathbf{n}) \leqslant 0 \\ &\delta\mathbf{u}_\tau < 0 \Rightarrow \sigma_\tau + \delta\sigma_\tau + \mathbf{f}(\sigma_\mathbf{n} + \delta\sigma_\mathbf{n}) = 0 \end{aligned} \tag{1.5}$$

2. A DIFFERENTIAL FORMULATION

We will now consider the quasistatic deformation of an elastic body which occupies the region $\bar{\mathbf{S}}$ in $\mathbf{R}^3$ with a regular boundary [4, p. 197] $\mathbf{L}$, $\bar{\mathbf{S}} = \mathbf{S} \cup \mathbf{L}$, with $\mathbf{L}$ divided into three mutually non-intersecting parts: $\mathbf{L} = \mathbf{L}_\mathbf{u} \cup \mathbf{L}_c \cup \mathbf{L}_\sigma$. On $\mathbf{L}_\mathbf{u}$ we are given kinematic boundary conditions, on $\mathbf{L}_\sigma$ the forces, and on

L_c the contact boundary conditions. We choose a reference system with respect to which the rigid support is fixed, in which the equations and boundary conditions of the problem are

$$\nabla \cdot \delta\tilde{\sigma} + \delta = 0, \quad \forall \mathbf{x} \in \mathbf{S} \tag{2.1}$$

$$\delta\tilde{\varepsilon} = \frac{1}{2}(\nabla\delta\mathbf{u} + \delta\mathbf{u}\nabla), \quad \delta\tilde{\sigma} = \tilde{\tilde{\Phi}} \cdot\cdot \delta\tilde{\varepsilon}, \quad \forall \mathbf{x} \in \bar{\mathbf{S}} \tag{2.2}$$

$$\delta\mathbf{u} = 0 \quad \text{on} \quad \mathbf{L}_u, \quad \delta\tilde{\sigma} \cdot \mathbf{n} = \delta\mathbf{P} \quad \text{on} \quad \mathbf{L}_\sigma \tag{2.3}$$

((2.1) are the equilibrium equations, (2.2) are the geometric and constitutive relations, and (2.3) are the kinematic and static boundary conditions).

To relations (2.1)–(2.3) we add the boundary conditions on the contact surface (1.5).

We now define the set

$$\mathbf{U} = \{\delta\mathbf{u} \in (\mathbf{C}^2(\bar{\mathbf{S}}))^2 \mid \delta\mathbf{u} = 0 \quad \text{on} \quad \mathbf{L}_u, \quad \delta\mathbf{u}_n = \delta\mathbf{u}_b = 0, \quad \delta\mathbf{u}_\tau \leqslant 0 \quad \text{on} \quad \mathbf{L}_c\}$$

A function $\delta\mathbf{u} \in \mathbf{U}$ which satisfies relations (1.5), (2.1)–(2.3) will be called a classical solution of the problem.

Under the given assumptions, at any time $\mathbf{t}$ the stress and strain fields can be related to the corresponding fields at time $\mathbf{t} + \delta\mathbf{t}$ after loading.

Note. It is assumed that the stressed state at the time of loading is known, with $\sigma_b = 0$ on the contact surface. It does not follow in the general case that the increment of this component $\delta\sigma_b$ after loading is zero. However, if the condition that the trajectories are rectilinear holds, we have $\delta\sigma_b = 0$. For instance, in the axisymmetric problem or the problem of a plane-stressed state, the domain of definition of the known quantities implies that the stress increment $\delta\sigma_b$ is zero.

3. A VARIATIONAL FORMULATION

Let $\delta\mathbf{u}$ be a classical solution of problem (1.5), (2.1)–(2.3), $\delta\mathbf{u} \in \mathbf{U}$. We first find the scalar product of the equilibrium equation (2.1) by the expression $\delta\mathbf{v} - \delta\mathbf{u}$, where $\delta\mathbf{v} \in \mathbf{U}$, and integrate over the region $\bar{\mathbf{S}}$, and apply the Ostrogradskii–Gauss theorem to the resulting expression. Using the constitutive and geometric relations, the boundary conditions, and the definition of the set $\mathbf{U}$, we obtain

$$\mathbf{A}(\delta\mathbf{u}, \delta\mathbf{v} - \delta\mathbf{u}) - \langle \mathbf{f}, \delta\mathbf{v} - \delta\mathbf{u} \rangle = \int_{\mathbf{L}_c} \delta\sigma_\tau(\delta\mathbf{v}_\tau - \delta\mathbf{u}_\tau)d\mathbf{L}_c, \quad \forall \delta\mathbf{v} \in \mathbf{U}$$

$$\mathbf{A}(\delta\mathbf{u}, \delta\mathbf{v} - \delta\mathbf{u}) = \int_{\mathbf{S}} \delta\tilde{\varepsilon}(\delta\mathbf{u}) \cdot\cdot \tilde{\tilde{\Phi}} \cdot\cdot \delta\tilde{\varepsilon}(\delta\mathbf{v} - \delta\mathbf{u})d\mathbf{S} \tag{3.1}$$

$$\langle \mathbf{f}, \delta\mathbf{v} - \delta\mathbf{u} \rangle = \int_{\mathbf{L}_\sigma} \delta\mathbf{P} \cdot (\delta\mathbf{v} - \delta\mathbf{u})d\mathbf{L}_\sigma + \int_{\mathbf{S}} \delta\mathbf{F} \cdot (\delta\mathbf{v} - \delta\mathbf{u})d\mathbf{S}$$

We then use the fact that on the contact surface $\mathbf{L}_c$

$$(\sigma_\tau + \delta\sigma_\tau + \mathbf{f}(\sigma_n + \delta\sigma_n))(\delta\mathbf{v}_\tau - \delta\mathbf{u}_\tau) \geqslant 0 \tag{3.2}$$

In fact, if $\delta\mathbf{u}_\tau = 0$ according to (1.5) the first multiplier is non-positive, and from the definition of the set $\mathbf{U}$ we have $\delta\mathbf{v}_\tau \leqslant 0$. But if $\delta\mathbf{u}_\tau < 0$, according to (1.5) the first multiplier is equal to zero and strict equality applies in (3.2).

We now add to the left- and right-hand sides of Eq. (3.1) the expression

$$\int_{\mathbf{L}_c} (\sigma_\tau + \mathbf{f}(\sigma_n + \delta\sigma_n))(\delta\mathbf{v}_\tau - \delta\mathbf{u}_\tau)d\mathbf{L}_c$$

We will introduce the notation $\mathbf{g} = \sigma_\tau + \mathbf{f}\sigma_n$, with $\mathbf{g} = 0$ on $\mathbf{L}_1$. Then, using (3.2), we obtain

$$\mathbf{A}(\delta\mathbf{u}, \delta\mathbf{v} - \delta\mathbf{u}) - \langle \mathbf{f}, \delta\mathbf{v} - \delta\mathbf{u} \rangle + \int_{\mathbf{L}_0} \mathbf{g}(\delta\mathbf{v}_\tau - \delta\mathbf{u}_\tau)d\mathbf{L}_0 + \int_{\mathbf{L}_c} \mathbf{f}\delta\sigma_n(\delta\mathbf{v}_\tau - \delta\mathbf{u}_\tau)d\mathbf{L}_c \geqslant 0, \quad \forall \delta\mathbf{v} \in \mathbf{U} \tag{3.3}$$

We have thus proved the following theorem.

Theorem 1. A classical solution of problem (1.5), (2.1)–(2.3) satisfies inequality (3.3).
Consider the following problem.

Problem 1. It is required to find $\delta\mathbf{u} \in \mathbf{U}$

$$A(\delta\mathbf{u},\delta\mathbf{v}-\delta\mathbf{u})-\langle\mathbf{f},\delta\mathbf{v}-\delta\mathbf{u}\rangle+\int_{L_0}\mathbf{g}(\delta\mathbf{v}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_0+$$
$$+\int_{L_c}\mathbf{f}\delta\sigma_n(\delta\mathbf{v}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_c\geqslant 0,\quad\forall\delta\mathbf{v}\in\mathbf{U}\tag{3.4}$$

Theorem 2. A solution $\delta\mathbf{u}$ of Problem 1 satisfies all the given kinematic boundary conditions in the relations of the differential problem (1.5) and (2.3), and the stress tensor defined in terms of $\delta\mathbf{u}$ by the geometric and constitutive relations (2.2) satisfies the equilibrium equations (2.1) and all the given boundary conditions in the forces in relations (1.5) and (2.3).

Proof. Suppose that inequality (3.4) has a solution $\delta\mathbf{u}$. If, in inequality (3.4), the element $\delta\mathbf{v} \in \mathbf{U}$ is taken first as $\delta\mathbf{u} + \mathbf{q} \in \mathbf{U}$, and then as $\delta\mathbf{u} - \mathbf{q} \in \mathbf{U}$, where the function $\mathbf{q} \in \mathbf{U}$ is finite in the region $\bar{S}$, then

$$A(\delta\mathbf{u},\mathbf{q})-\int_{S}\delta\mathbf{F}\cdot\mathbf{q}dS=0$$

It follows from this equation, the relation between $\delta\mathbf{u}$ and the strains and stresses (relations (2.2)), and the arbitrary choice of $\mathbf{q}$ that the equilibrium equation (2.1) holds. We find the scalar product of Eq. (2.1) and the vector $\delta\mathbf{u} - \delta\mathbf{v}$, where $\delta\mathbf{v}, \delta\mathbf{u} \in \mathbf{U}$, integrate the resulting expression over the region $\bar{S}$ and apply the Ostrogradskii–Gauss theorem. We then subtract the resulting relation from inequality (3.4). We finally obtain the inequality

$$\int_{L_0}(\delta\mathbf{v}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_0+\int_{L_c}\mathbf{f}\delta\sigma_n(\delta\mathbf{v}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_c-\int_{L_\sigma}\delta\mathbf{P}\cdot(\delta\mathbf{v}-\delta\mathbf{u})d\mathbf{L}_\sigma+$$
$$+\int_{L_\sigma}\mathbf{n}\cdot\delta\tilde{\sigma}(\delta\mathbf{v}-\delta\mathbf{u})d\mathbf{L}_\sigma+\int_{L_c}\delta\sigma_\tau(\delta\mathbf{v}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_c\geqslant 0,\quad\forall\delta\mathbf{v}\in\mathbf{U}\tag{3.5}$$

We choose an element $\delta\mathbf{v} \in \mathbf{U}$ for which $\delta\mathbf{u} = \delta\mathbf{v}$ on $\mathbf{L}\backslash\mathbf{L}_\sigma$. Then inequality (3.5) will take the form

$$\int_{L_\sigma}(\delta\mathbf{P}-\mathbf{n}\cdot\delta\tilde{\sigma})\cdot(\delta\mathbf{v}-\delta\mathbf{u})d\mathbf{L}_\sigma\geqslant 0$$

whence it follows, since $\delta\mathbf{v}$ is arbitrary on $\mathbf{L}_\sigma$, that the static boundary conditions in (2.3) are satisfied.
Now, taking into account the definition of $\mathbf{g}$ and the fact that static boundary conditions are satisfied, we can write inequality (3.5) in the form

$$\int_{L_c}(\sigma_\tau+\delta\sigma_\tau+\mathbf{f}(\sigma_n+\delta\sigma_n))(\delta\mathbf{v}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_c\geqslant 0,\quad\forall\delta\mathbf{v}\in\mathbf{U}\tag{3.6}$$

The points of the contact surface where the equation $\delta\mathbf{u}_\tau = 0$ is satisfied form a region $\mathbf{L}_0^*$. Since $\delta\mathbf{v}_\tau$ is non-positive on $\mathbf{L}_0^*$, and was chosen arbitrarily from inequality (3.6), it follows that

$$(\sigma_\tau+\delta\sigma_\tau)+\mathbf{f}(\sigma_n+\delta\sigma_n)\leqslant 0\quad\text{on}\quad\mathbf{L}_0^*$$

On the other part of the contact surface $\mathbf{L}^*_1$ $\delta\mathbf{u}_\tau < 0$. Then, since $\delta\mathbf{v}_\tau$ was arbitrarily chosen, it follows from inequality (3.6) that

$$(\sigma_\tau+\delta\sigma_\tau)+\mathbf{f}(\sigma_n+\delta\sigma_n)=0\quad\text{on}\quad\mathbf{L}_1^*$$

The kinematic boundary conditions $\delta\mathbf{u} = 0$ on $\mathbf{L}_u$ and $\delta\mathbf{u}_n = \delta\mathbf{u}_b = 0$ on $\mathbf{L}_c$ follow from the definition of the set $\mathbf{U}$. This proves the theorem.
A solution of Problem 1 in the wider class of functions

$$\mathbf{W} = \{\delta\mathbf{u} \in (\mathbf{H}^1(\mathbf{S}))^3 \mid \delta\mathbf{u} = 0 \text{ on } \mathbf{L}_u, \quad \delta\mathbf{u}_n = \delta\mathbf{u}_b = 0, \quad \delta\mathbf{u}_\tau \leq 0 \text{ on } \mathbf{L}_c\}$$

where $(\mathbf{H}^1(\mathbf{S}))^3$ is a Sobolev space [4] will be called a generalized solution. Here and below function values on the boundary are understood as the values of the trace operator [4, pp. 48–51].

The analysis of generalized solutions of Problem 1 is complicated by the fact that the operator which sets the element $\delta\mathbf{u} \in \mathbf{W}$ in correspondence with the distribution of the contact pressure $\delta\sigma_n$ on the contact surface $\mathbf{L}_c$ is undefined [2, p. 148].

4. OPERATOR OF THE INFLUENCE OF RELATIVE SHEARS ON THE CONTACT PRESSURE

Suppose that the tangential displacements $\delta\mathbf{y}_\tau$, defined by the trace operator [4]

$$\gamma_\tau\colon (\mathbf{H}^1(\mathbf{S}))^3 \to \mathbf{H}^{1/2}(\mathbf{L}_c)$$

are given on the contact surface $\delta\mathbf{y}_\tau$.

Using the given tangential displacements $\delta\mathbf{y}_\tau$, which are the trace of elements of the set $\mathbf{W}$, we define the set

$$\mathbf{Z} = \{\delta\mathbf{v} \in \mathbf{W} \mid \delta\mathbf{v}_\tau = \delta\mathbf{y}_\tau = \gamma_\tau(\delta\mathbf{y}) \text{ on } \mathbf{L}_c\}$$

We now consider the boundary-value problem (2.1)–(2.3) with kinematic boundary conditions

$$\delta\mathbf{u}_n = \delta\mathbf{u}_b = 0, \quad \delta\mathbf{u}_\tau = \delta\mathbf{y}_\tau \text{ on } \mathbf{L}_c \tag{4.1}$$

The following problem is the equivalent of problem (2.1)–(2.3), (4.1) for smooth functions in a variational formulation.

Problem 2. It is required to find $\delta\mathbf{u} \in \mathbf{Z}$

$$\mathbf{A}(\delta\mathbf{u}, \delta\mathbf{v} - \delta\mathbf{u}) = \langle \mathbf{f}, \delta\mathbf{v} - \delta\mathbf{u} \rangle, \quad \forall \delta\mathbf{v} \in \mathbf{Z} \tag{4.2}$$

We will prove this statement. Let $\delta\mathbf{u}$ be a solution of problem (2.1)–(2.3), (4.1). Then relation (4.2) is obtained by repeating the proof of relation (3.1), allowing for the fact that $\delta\mathbf{v}_\tau = \delta\mathbf{u}_\tau = \delta\mathbf{y}_\tau$ on $\mathbf{L}_c$.

We now prove the inverse. Let $\delta\mathbf{u}$ be a solution of Problem 2. Take $\delta\mathbf{v} = \delta\mathbf{u} + \mathbf{q}$, where the function $\mathbf{q}$ is finite in the region $\bar{\mathbf{S}}$. Then the equilibrium equation (2.1) follows from relation (4.2) is the deformations and stresses on $\delta\mathbf{u}$ are defined by relations (2.2). Taking the scalar product of Eq. (2.1) by $\delta\mathbf{v} - \delta\mathbf{u}$. $\delta\mathbf{v} \in Z$, integrating it over the region $\bar{\mathbf{S}}$, applying the Ostrogradskii–Gauss theorem and then subtracting the resulting relation from (4.2), we have

$$0 = \int_{\mathbf{L}_\sigma} \delta\mathbf{P} \cdot (\delta\mathbf{v} - \delta\mathbf{u}) d\mathbf{L}_\sigma - \int_{\mathbf{L}_\sigma} \mathbf{n} \cdot \delta\tilde{\sigma} \cdot (\delta\mathbf{v} - \delta\mathbf{u}) d\mathbf{L}_\sigma, \quad \forall \delta\mathbf{v} \in \mathbf{Z}$$

Since the choice of $\delta\mathbf{v}$ was arbitrary, the last equation gives conditions in forces on the surface $\mathbf{L}_\sigma$.

The kinematic boundary conditions on the surfaces $\mathbf{L}_u$ and $\mathbf{L}_c$ follow from the definition of the set $\mathbf{Z}$.

We will need an equivalent statement of Problem 2 in the form of a minimization of a quadratic functional.

We define the space

$$\mathbf{X} = \{\delta\mathbf{v} \in (\mathbf{H}^1(\mathbf{S}))^3 \mid \delta\mathbf{v} = 0 \text{ on } \mathbf{L}_u\}$$

and introduce the norm

$$\|\delta\mathbf{v}\|_X = (\delta\mathbf{v}, \delta\mathbf{v})^{1/2}, \quad (\delta\mathbf{v}, \delta\mathbf{v}) = \int_{\mathbf{S}} \delta\tilde{\varepsilon}(\delta\mathbf{v}) \cdot\cdot \tilde{\tilde{\Phi}} \cdot\cdot \delta\tilde{\varepsilon}(\delta\mathbf{v}) d\mathbf{S} \tag{4.3}$$

where $(\delta\mathbf{v}, \delta\mathbf{v})$ is the scalar product in the space $\mathbf{X}$ [2, 4].

We also define the space

$$\mathbf{Z}_0 = \{\delta\upsilon \in \mathbf{X} \mid \delta\upsilon = 0 \text{ on } \mathbf{L}_c\}$$

which is a subspace of the space $\mathbf{X}$.

Any element $x \in \mathbf{X}$ can be represented uniquely in the form

$$\mathbf{x} = \mathbf{y} + \mathbf{z}, \text{ where } y \in \mathbf{Z}_0, \mathbf{z} \in \mathbf{X}, \mathbf{z} \perp \mathbf{Z}_0 \tag{4.4}$$

[9, Theorem 6.4.1]

Consider two arbitrary elements $\mathbf{a}, \mathbf{b} \in \mathbf{X}$ for which $\mathbf{a} = \mathbf{b}$ on $\mathbf{L_c}$. Using (4.4), we will represent these elements as follows:

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_\perp, \mathbf{b} = \mathbf{b}_0 + \mathbf{b}_\perp$$
$$\mathbf{a}_0, \mathbf{b}_0 \in \mathbf{Z}_0,\ \mathbf{a}_\perp, \mathbf{b}_\perp \in \mathbf{X},\ \mathbf{a}_\perp, \mathbf{b}_\perp \perp \mathbf{Z}_0$$

We carry out the transformations

$$\mathbf{a} = \mathbf{b} + (\mathbf{a} - \mathbf{b}) = \mathbf{b}_0 + \mathbf{b}_\perp + (\mathbf{a} - \mathbf{b}) =$$
$$= b_\perp + [\mathbf{b}_0 + (\mathbf{a} - \mathbf{b})] = \mathbf{b}_\perp + \mathbf{c}_0\ (\mathbf{c}_0 = [\mathbf{b}_0 + (\mathbf{a} - \mathbf{b})] \in \mathbf{Z}_0) \tag{4.5}$$

Since the expansion (4.5) is unique, we obtain $\mathbf{a}_\perp = \mathbf{b}_\perp$.

Thus, any element $\delta\boldsymbol{v} \in \mathbf{X}$ for which $\delta\boldsymbol{v} = \delta\mathbf{y}$ on $\mathbf{L_c}$ ($\delta\mathbf{y} \in \mathbf{X}$) can be represented uniquely in the form

$$\delta\boldsymbol{v} = \delta\boldsymbol{v}_0 + \delta\mathbf{w} \tag{4.6}$$

where

$$\delta\boldsymbol{v}_0 \in \mathbf{Z}_0,\ \delta\mathbf{w} \in \mathbf{X},\ \delta\mathbf{w} \perp \mathbf{Z}_0,\ \delta\mathbf{w} = \delta\mathbf{y} \text{ on } \mathbf{L_c}$$

where $\delta\mathbf{w}$ is independent of the choice of $\delta\boldsymbol{v}$.

We can use Eq. (4.6) to expand the solution of Problem 2, since $\delta\mathbf{u} \in \mathbf{X}$. We have

$$\delta\mathbf{u} = \delta\mathbf{u}_0 + \delta\mathbf{w}$$

where

$$\delta\mathbf{u}_0 \in \mathbf{X}_0,\ \delta\mathbf{w} \in \mathbf{X},\ \delta\mathbf{w} \perp \mathbf{X}_0,\ \mathbf{w}_\tau = \mathbf{y}_\tau,\ \mathbf{w}_\mathbf{n} = \mathbf{w}_0 = 0 \text{ on } \mathbf{L_c}$$

Applying this expansion also to elements $\delta\boldsymbol{v} \in \mathbf{Z}$ in (4.2), we find that

$$\mathbf{A}\ (\delta\mathbf{w} + \delta\mathbf{u}_0, \delta\boldsymbol{v}_0 - \delta\mathbf{u}_0) = \langle\mathbf{f}, \delta\boldsymbol{v}_0 - \delta\mathbf{u}_0\rangle$$

whence for $\delta\boldsymbol{v}_0 = 0$ we have

$$\mathbf{A}\ (\delta\mathbf{u}_0, \delta\mathbf{u}_0) = \langle\mathbf{f}, \delta\mathbf{u}_0\rangle$$

Using (4.6) and the last equation, we can reformulate Problem 2 as follows.

Problem 2a. It is required to find $\delta\boldsymbol{v}_0 \in \mathbf{Z}_0$

$$\mathbf{A}\ (\delta\mathbf{u}_0, \delta\boldsymbol{v}_0) = \langle\mathbf{f}, \delta\boldsymbol{v}_0\rangle,\ \forall\ \delta\boldsymbol{v}_0 \in \mathbf{Z}_0,\ \delta\mathbf{u} = \delta\mathbf{u}_0 + \delta\mathbf{w},\ \delta\mathbf{w} \in \mathbf{Z}, \delta\mathbf{w} \perp \mathbf{Z}_0 \tag{4.7}$$

The contact pressure can be found in terms of Lagrange multipliers. We will omit the constraint on the normal component of the displacements in the contact region $\delta\mathbf{u_n} = 0$ when determining an admissible set of displacements $\mathbf{Z}_0$. This can be done with a Lagrange multiplier which, as we shall show, is the same as the increment of the normal component of the reaction of the rigid support during loading.

We then define the functional

$$\mathbf{J}\ (\delta\boldsymbol{v}_0) = \frac{1}{2}\mathbf{A}\ (\delta\boldsymbol{v}_0, \delta\boldsymbol{v}_0) - \langle\mathbf{f}, \delta\boldsymbol{v}_0\rangle$$

and consider an equivalent problem to Problem 2a.

Problem 3. It is required to find $\delta \mathbf{u}_0 \in \mathbf{Z}_0$

$$J(\delta \mathbf{u}_0) \leq J(\delta \boldsymbol{v}_0), \quad \forall \delta \boldsymbol{v}_0 \in \mathbf{Z}_0, \quad \delta \mathbf{u} = \delta \mathbf{u}_0 + \delta \mathbf{w}, \quad \delta \mathbf{w} \in \mathbf{Z}, \quad \delta \mathbf{w} \perp \mathbf{Z}_0 \tag{4.8}$$

The solution of Problem 3 is unchanged when the expression $\mathbf{A}(\delta \mathbf{w}, \delta \boldsymbol{v}_0)$ is added to the functional $\mathbf{J}(\delta \boldsymbol{v}_0)$, since $\delta \boldsymbol{v}_0 \in \mathbf{Z}_0$ and $\delta \mathbf{w} \perp \mathbf{Z}_0$. We can therefore write a dual formulation of Problem 3 as follows [10].

Problem 4. It is required to find

$$\sup_{\mathbf{p}^* \in \mathbf{H}_*^{1/2}(\mathbf{L}_c)} \; \inf_{\delta \boldsymbol{v}_0 \in \mathbf{Y}_0} \Phi(\delta \boldsymbol{v}_0, \mathbf{p}^*) \tag{4.9}$$

$$\mathbf{Y}_0 = \{\delta \boldsymbol{v} \in \mathbf{X} \mid \delta \boldsymbol{v}_\tau = \delta \boldsymbol{v}_b = 0 \quad \text{on} \quad \mathbf{L}_c\}$$

$$\Phi(\delta \boldsymbol{v}_0, \mathbf{p}^*) = \mathbf{J}(\delta \boldsymbol{v}_0) + \mathbf{A}(\delta \mathbf{w}, \delta \boldsymbol{v}_0) - \left\langle \mathbf{p}^*, \delta \boldsymbol{v}_{0n} \right\rangle$$

$$\delta \boldsymbol{v}_{0n} \equiv \gamma_n(\delta \boldsymbol{v}_0), \quad \gamma_n : (\mathbf{H}^1(\mathbf{S}))^3 \to \mathbf{H}^{1/2}(\mathbf{L}_c)$$

where $\mathbf{H}_*^{1/2}(\mathbf{L}_c)$ is the conjugate space to $\mathbf{H}^{1/2}(\mathbf{L}_c)$ and γ_n is the trace operator defining the normal component of displacements on $\mathbf{L}_c$.

Now suppose that $(\delta \mathbf{u}_0, \mathbf{q}^*) \in \mathbf{Y}_0 \times \mathbf{H}_*^{1/2}(\mathbf{L}_c)$ is a solution of Problem 4. We will show that the first component $\delta \mathbf{u}_0$ is a solution of Problem 3, and the second component $\mathbf{q}^*$ is the increment of the normal component of the contact pressure.

For a saddle point we have

$$\frac{\partial}{\partial \alpha} \Phi(\delta \mathbf{u}_0 + \alpha \delta \boldsymbol{v}_0, \mathbf{q}^*)\Big|_{\alpha=0} = 0, \quad \forall \delta \boldsymbol{v}_0 \in \mathbf{Y}_0$$

$$\frac{\partial}{\partial \alpha} \Phi(\delta \mathbf{u}_0, \mathbf{q}^* + \alpha \mathbf{p}^*)\Big|_{\alpha=0} = 0, \quad \forall \mathbf{p}^* \in \mathbf{H}_*^{1/2}(\mathbf{L}_c); \quad \alpha \in \mathbf{R}$$

After computing the derivatives we obtain

$$\mathbf{A}(\delta \mathbf{w} + \delta \mathbf{u}_0, \delta \boldsymbol{v}_0) - \left\langle \mathbf{f}, \delta \boldsymbol{v}_0 \right\rangle - \left\langle \mathbf{q}^*, \delta \boldsymbol{v}_{0n} \right\rangle = 0, \quad \forall \delta \boldsymbol{v}_0 \in \mathbf{Y}_0 \tag{4.10}$$

$$\left\langle \mathbf{p}^*, \delta \mathbf{u}_{0n} \right\rangle = 0, \quad \forall \mathbf{p}^* \in \mathbf{H}_*^{1/2}(\mathbf{L}_c) \tag{4.11}$$

From relation (4.11), since the choice of $\mathbf{p}^*$ was arbitrary, we have $\delta \boldsymbol{v}_{0n} = 0$ on $\mathbf{L}_c$. Hence, $\delta \mathbf{u}_0 \in \mathbf{Z}_0$. We now consider relation (4.10). We have $\mathbf{Z}_0 \subset \mathbf{Y}_0$, and thus relation (4.10) holds for any $\delta \boldsymbol{v}_0 \in \mathbf{Z}_0$. Then

$$\mathbf{A}(\delta \mathbf{u}_0, \delta \boldsymbol{v}_0) - \left\langle \mathbf{f}, \delta \boldsymbol{v}_0 \right\rangle = 0, \quad \forall \delta \boldsymbol{v}_0 \in \mathbf{Z}_0$$

Thus we have proved that $\delta \mathbf{u}_0$ is a solution of Problem 3.

We will determine the meaning of a Lagrange multiplier in terms of mechanics on the assumption that $\delta \mathbf{u}_0 + \delta \mathbf{w}$ is a regular function. Then in (4.10), taking $\delta \boldsymbol{v}_0$ as finite functions on $\bar{s}$ and using the relation between the displacements, strains and stresses (2.2), we obtain the equilibrium equation in the form

$$\nabla \cdot \delta \tilde{\sigma}(\delta \mathbf{u}_0 + \delta \mathbf{w}) + \delta \mathbf{F} = 0 \tag{4.12}$$

Taking the scalar product of Eq. (4.12) by an arbitrary element $\delta \boldsymbol{v}_0 \in \mathbf{Y}_0$, integrating over the region $\bar{s}$ and applying the Ostrogradskii–Gauss theorem, we obtain an equation which we subtract from Eq. (4.10) and obtain the expression

$$\left\langle \mathbf{q}^*, \delta \boldsymbol{v}_{n0} \right\rangle = \int_{L_c} \mathbf{n} \cdot \delta \tilde{\sigma}(\delta \mathbf{u}_0 + \delta \mathbf{w}) \cdot \mathbf{n} \delta \boldsymbol{v}_{0n} d\mathbf{L}_c, \quad \forall \delta \boldsymbol{v}_0 \in \mathbf{Y}_0 \tag{4.13}$$

which gives the meaning of the Lagrange multiplier in terms of mechanics.

We will define the Hilbert space

$$\mathbf{H}=\{\delta\boldsymbol{\upsilon}\in\mathbf{X}\mid\delta\boldsymbol{\upsilon}_n=\delta\boldsymbol{\upsilon}_b=0\ \text{ on }\ \mathbf{L}_c\}$$

Theorem 3. If $\mathbf{Y}_0\perp\mathbf{H}$, the increment of contact pressure during loading is independent of the size of the relative shears.

The theorem is proved by repeating the derivation of (4.13) taking into account the equation $\mathbf{A}(\delta\mathbf{w},\delta\boldsymbol{\upsilon}_0)=0$, which follows from the fact that the spaces are orthogonal.

For contact problems to do with plates, orthogonality with respect to the energy norm of the spaces used is a consequence of the fact that the membrane displacements which determine the tangential displacements in the contact region are independent of the displacements normal to the surface of the plate. Other cases in which the contact pressure is independent of the friction and cohesive forces are discussed in [5]. This class of problem has been studied in detail in [5–7] and elsewhere.

Thus, the method can be used to determine the contact pressure in the sense of Eq. (4.13) for given relative shears. The operator which sets the increment of contact pressure in correspondence with the tangential shears $\delta\mathbf{w}_\tau$ and $\mathbf{L}_c$ is thereby determined

$$\mathbf{p}:\delta\mathbf{w}_\tau\to\mathbf{q}^*\in\mathbf{H}_*^{1/2}(\mathbf{L}_c) \tag{4.14}$$

where $\mathbf{q}^*$ is a linear functional.

We will now investigate the properties of the functional $\mathbf{q}^*$. The condition $\delta\mathbf{w}_\tau=0$ corresponds to the functional $\mathbf{q}_0^*$ defined by relation (4.10)

$$\left\langle\mathbf{q}_0^*,\ \delta\boldsymbol{\upsilon}_{0n}\right\rangle=\mathbf{A}(\delta\mathbf{u}_0,\ \delta\boldsymbol{\upsilon}_0)-\left\langle\mathbf{f},\ \delta\boldsymbol{\upsilon}_0\right\rangle,\quad\forall\delta\boldsymbol{\upsilon}_0\in\mathbf{Y}_0 \tag{4.15}$$

where $\delta\mathbf{u}_0$ is a solution of Problem 3.

We will now consider an arbitrary distribution of shears $\delta\mathbf{w}_\tau$ on $\mathbf{L}_c$ and use (4.10) to determine the functional $\mathbf{q}^*$ responsible for the contact pressure distribution

$$\left\langle\mathbf{q}^*,\ \delta\boldsymbol{\upsilon}_{0n}\right\rangle=\mathbf{A}(\delta\mathbf{u}_0+\ \delta\mathbf{w},\ \delta\boldsymbol{\upsilon}_0)-\left\langle\mathbf{f},\ \delta\boldsymbol{\upsilon}_0\right\rangle,\quad\forall\delta\boldsymbol{\upsilon}_0\in\mathbf{Y}_0 \tag{4.16}$$

where, as before, $\delta\mathbf{u}_0$ is a solution of Problem 3.

Subtracting Eq. (4.15) from (4.16), we have

$$\left\langle\mathbf{q}_w^*,\ \delta\boldsymbol{\upsilon}_{0n}\right\rangle=\mathbf{A}(\delta\mathbf{w},\ \delta\boldsymbol{\upsilon}_0),\quad\forall\delta\boldsymbol{\upsilon}_0\in\mathbf{Y}_0;\ \ \mathbf{q}_w^*=\mathbf{q}^*-\mathbf{q}_0^* \tag{4.17}$$

Thus, the operator which determines the contact pressure can be represented in the form of a sum of two linear functionals

$$\mathbf{q}^*=\mathbf{q}_0^*+\mathbf{q}_w^*$$

Since the mapping $\delta\mathbf{w}\to\mathbf{q}_w^*$ is linear (relation (4.17)), the linear functional $\mathbf{q}_w^*$ can be represented by a bilinear form, which we will denote by $\mathbf{p}_w$. The one-to-one correspondence (4.6) between $\delta\mathbf{w}_\tau$ on $\mathbf{L}_c$ and $\mathbf{w}\in\mathbf{Z}$ and relation (4.17) enables us to define $\mathbf{p}_w$ as follows:

$$\mathbf{p}_w(\cdot,\ \cdot):\mathbf{H}^{1/2}(\mathbf{L}_c)\times\mathbf{H}^{1/2}(\mathbf{L}_c)\to\mathbf{R}$$

Hence we have proved that there is a linear operator which determines the contact pressure from the relative shears. We will use this operator to rewrite the last integral in (3.4)

$$\int_{L_c}\mathbf{f}\,\delta\alpha_n(\delta\boldsymbol{\upsilon}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_c=\mathbf{f}\Big[\left\langle\mathbf{q}_0^*,\ \delta\boldsymbol{\upsilon}_\tau-\delta\mathbf{u}_\tau\right\rangle+\mathbf{p}_w(\delta\mathbf{u}_\tau,\ \delta\boldsymbol{\upsilon}_\tau-\delta\mathbf{u}_\tau)\Big] \tag{4.18}$$

The existence and uniqueness of a solution of problems of the type (4.10), (4.11) has been considered earlier [10, Proposition 5.2, p. 74]. In the case of elastic materials, a solution of the problem exists and its first component $\delta\mathbf{u}_0$ is unique. Using the regularization method of [11] and taking the limit as the regularizing addition tends to zero, it can be proved that the Lagrange multiplier is unique.

Thus in this section we have proved that there is a linear operator which sets the relative shears on the contact surface in correspondence with the contact pressure increment.

5. PROOF OF EXISTENCE AND UNIQUENESS OF THE GENERALIZED SOLUTION

We will rewrite Problem 1 to take Eq. (4.18) into account, as follows.

Problem 1a. It is required to find $\delta\mathbf{u} \in \mathbf{W}$

$$\mathbf{B}(\delta\mathbf{u},\ \delta\boldsymbol{v}-\delta\mathbf{u})-\left\langle \mathbf{s},\ \delta\boldsymbol{v}-\delta\mathbf{u}\right\rangle \geqslant 0, \quad \forall\delta\boldsymbol{v}\in\mathbf{W} \tag{5.1}$$

$$\mathbf{B}(\delta\mathbf{u},\ \delta\boldsymbol{v}-\delta\mathbf{u})=\mathbf{A}(\delta\mathbf{u},\ \delta\boldsymbol{v}-\delta\mathbf{u})+\mathbf{f}\mathbf{p}_{\mathbf{w}}(\delta\mathbf{u}_\tau,\ \delta\boldsymbol{v}_\tau-\delta\mathbf{u}_\tau)$$

$$\left\langle \mathbf{s},\ \delta\boldsymbol{v}-\delta\mathbf{u}\right\rangle=\int\limits_{\mathbf{L}_0}\mathbf{g}(\delta\boldsymbol{v}_\tau-\delta\mathbf{u}_\tau)d\mathbf{L}_0-\left\langle \mathbf{f},\ \delta\boldsymbol{v}-\delta\mathbf{u}\right\rangle+\mathbf{f}\left\langle \mathbf{q}_0^*,\ \delta\boldsymbol{v}_\tau-\delta\mathbf{u}_\tau\right\rangle$$

We will now take the generalized solution to mean the solution of Problem 1a.

The existence of the operator $\mathbf{p}_\mathbf{w}$ was proved above. Before the question posed in this section can be answered, we need to investigate the properties of this operator.

We will define a norm in the space $\mathbf{H}^{1/2}(\mathbf{L}_c)$, which is the region of values of the trace operator γ_τ with domain of definition $\mathbf{H}$

$$\left\|\delta\mathbf{z}\right\|_{\mathbf{H}^{1/2}(\mathbf{L}_c)}=\inf_{\substack{\delta\boldsymbol{v}\in\mathbf{H},\\ \gamma_\tau(\delta\boldsymbol{v})=\delta\mathbf{z}\text{ on }\mathbf{L}_c}}(\delta\boldsymbol{v},\ \delta\boldsymbol{v})_\mathbf{X}^{1/2}=(\delta\mathbf{w},\ \delta\mathbf{w})_{\substack{\mathbf{X}\\ \delta\mathbf{w}\perp\mathbf{Z}_0\\ \gamma_\tau(\delta\mathbf{w})=\delta\mathbf{z}\text{ on }\mathbf{L}_c}}^{1/2} \tag{5.2}$$

Relation (4.17) implies that

$$\left\|\mathbf{q}_\mathbf{w}^*\right\|_{\mathbf{H}_*^{1/2}(\mathbf{L}_c)}=\left\|\mathbf{p}_\mathbf{w}^*(\delta\mathbf{w}_\tau,\ \cdot)\right\|_{\mathbf{H}_*^{1/2}(\mathbf{L}_c)}=\sup\mathbf{A}(\delta\mathbf{w},\ \delta\boldsymbol{v}_\perp)\leqslant$$

$$\leqslant\left\|\delta\mathbf{w}\right\|_\mathbf{X}\leqslant\left\|\delta\mathbf{u}\right\|_\mathbf{X},\quad\forall\delta\mathbf{u}\in\mathbf{H},\quad\gamma_\tau(\delta\mathbf{u})=\delta\mathbf{w}_\tau\text{ on }\mathbf{L}_c \tag{5.3}$$

where sup is taken over $\delta\boldsymbol{v}_\perp\in\mathbf{Y}_0$, $\|\delta\boldsymbol{v}_\perp\|_\mathbf{X}=1$, $\delta\mathbf{w}\in\mathbf{H}$, $\delta\mathbf{w}\perp\mathbf{Z}_0$, $\gamma_\tau(\delta\mathbf{w})=\delta\mathbf{w}_\tau$ on $\mathbf{L}_c$.

From inequality (5.3) and the definition of the norm (5.2), we obtain the inequalities

$$\mathbf{p}_\mathbf{w}(\delta\mathbf{u}_\tau,\ \delta\mathbf{u}_\tau)\geqslant-|\mathbf{p}_\mathbf{w}(\delta\mathbf{u}_\tau,\ \delta\mathbf{u}_\tau)|\geqslant-\left\|\mathbf{q}_\mathbf{w}^*\right\|_{\mathbf{H}_*^{1/2}(\mathbf{L}_c)}\left\|\mathbf{u}_\tau\right\|_{\mathbf{H}_*^{1/2}(\mathbf{L}_c)}\geqslant$$

$$\geqslant-\left\|\delta\mathbf{u}\right\|_\mathbf{X}^2,\quad\forall\delta\mathbf{u}\in\mathbf{H}:\gamma\delta\mathbf{u}=\delta\mathbf{u}_\tau\text{ on }\mathbf{L}_c \tag{5.4}$$

Relation (5.4) and the condition $0<\mathbf{f}<1$ imply that the operator $\mathbf{B}$ is coercive

$$\mathbf{B}(\delta\mathbf{u},\ \delta\mathbf{u})=\mathbf{A}(\delta\mathbf{u},\ \delta\mathbf{u})+\mathbf{f}\mathbf{p}_\mathbf{w}(\delta\mathbf{u}_\tau,\ \delta\mathbf{u}_\tau)\geqslant\left\|\delta\mathbf{u}\right\|_\mathbf{X}^2-\mathbf{f}\left\|\delta\mathbf{u}\right\|_\mathbf{X}^2=$$

$$=(1-\mathbf{f})\left\|\delta\mathbf{u}\right\|_\mathbf{X}^2,\quad\forall\delta\mathbf{u}\in\mathbf{H} \tag{5.5}$$

We will inequality (5.3) to prove that $\mathbf{B}$ is bounded

$$\mathbf{B}(\delta\mathbf{u},\ \delta\boldsymbol{v})\leqslant\left\|\delta\mathbf{u}\right\|_\mathbf{X}\left\|\delta\boldsymbol{v}\right\|_\mathbf{X}+\mathbf{f}\left\|\mathbf{q}_\mathbf{w}^*\right\|_{\mathbf{H}_*^{1/2}(\mathbf{L}_c)}\left\|\delta\boldsymbol{v}_\tau\right\|_{\mathbf{H}_*^{1/2}(\mathbf{L}_c)}\leqslant$$

$$\leqslant\left\|\delta\mathbf{u}\right\|_\mathbf{X}\left\|\delta\boldsymbol{v}\right\|_\mathbf{X}+\mathbf{f}\left\|\delta\mathbf{u}\right\|_\mathbf{X}\left\|\delta\boldsymbol{v}\right\|_\mathbf{X}=(1+\mathbf{f})\left\|\delta\mathbf{u}\right\|_\mathbf{X}\left\|\delta\boldsymbol{v}\right\|_\mathbf{X},\quad\forall\delta\mathbf{u},\ \delta\boldsymbol{v}\in\mathbf{H} \tag{5.6}$$

There is a theorem [12] which states that if $\mathbf{B}$ is a coercive bilinear form on $\mathbf{H}$, $\mathbf{W}\subset\mathbf{H}$ is a closed convex set and $\mathbf{h}\in\mathbf{H}^*$ ($\mathbf{H}^*$ is the conjugate space to $\mathbf{H}$), then there is a unique solution of the following problem.

It is required to find $\delta\mathbf{u}\in\mathbf{W}$

$$\mathbf{B}(\delta\mathbf{u},\ \delta\boldsymbol{v}-\delta\mathbf{u})-\left\langle \mathbf{h},\ \delta\boldsymbol{v}-\delta\mathbf{u}\right\rangle\geqslant 0,\quad\forall\delta\boldsymbol{v}\in\mathbf{W}$$

We will compare the conditions of this theorem with those of the problem solved above. The operator **B** is linear in each of its arguments. Since (5.6) is bounded, **B** is continuous. Hence it is a bilinear form **H**. That **B** is coercive follows from inequality (5.5). Suppose that the volume and surface forces satisfy the condition $\mathbf{h} \in \mathbf{H}^*$. The convexity and closure of the set **W** are obvious. Thus the conditions of the above theorem are satisfied and we have proved the following theorem.

Theorem 4.1. Problem 1a has a unique generalized solution.

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